Sensitivity to initial conditions at bifurcations in one-dimensional nonlinear maps: rigorous nonextensive solutions

F. Baldovin[†] and A. Robledo[‡] *
[†]*Centro Brasileiro de Pesquisas Físicas*
Rua Xavier Sigaud 150,22290-180 Rio de Janeiro – RJ, Brazil
[‡]*Instituto de Física, Universidad Nacional Autónoma de México*
Apartado Postal 20-364, México 01000 D.F., Mexico.

September 10, 2002

Abstract

Using the Feigenbaum renormalization group (RG) transformation we work out exactly the dynamics and the sensitivity to initial conditions for unimodal maps of nonlinearity $\zeta > 1$ at both their pitchfork and tangent bifurcations. These functions have the form of q-exponentials as proposed in Tsallis' generalization of statistical mechanics. We determine the q-indices that characterize these universality classes and perform for the first time the calculation of the q-generalized Lyapunov coefficient λ_q. The pitchfork and the left-hand side of the tangent bifurcations display weak insensitivity to initial conditions, while the right-hand side of the tangent bifurcations presents a 'super-strong' (faster than exponential) sensitivity to initial conditions. We corroborate our analytical results with *a priori* numerical calculations.

PACS numbers: 05.10.Cc, 05.45.Ac, 05.90.+m

The nonextensive generalization [1] of the canonical statistical mechanics has raised interest in testing its applicability in several suggested circumstances in a variety of physical systems [1]. A class of problems where this issue has been much studied recently is the dynamical behavior of nonlinear iterated maps under critical conditions [2, 3, 4, 5, 6]. Such is the case of the bifurcation points associated to deterministic chaos in simple nonlinear dissipative maps, like those occurring in the logistic map and its generalization to nonlinearity $\zeta > 1$. These types of critical states provide an exceptional opportunity to examine explicitly the underlying mathematical structure and physical implications of their

*E-mail addresses: baldovin@cbpf.br, robledo@fisica.unam.mx

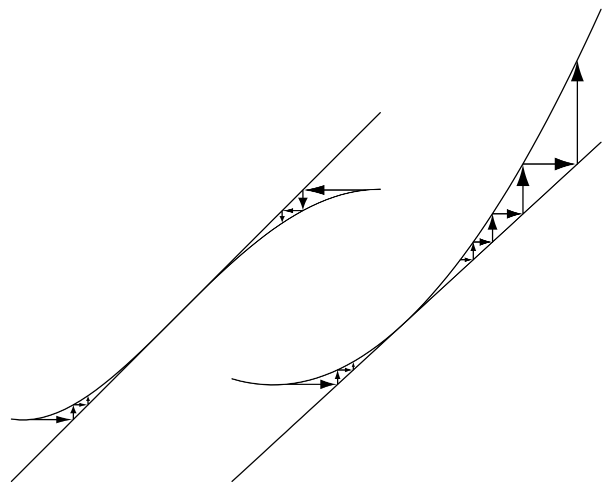


Figure 1: Schematic form of $f^{(n)}$ at the a) pitchfork and b) tangent bifurcations.

scale-invariant, power-law properties. As it is well-known the period-doubling and intermittency routes to chaos are based on the pitchfork and tangent bifurcations, respectively [7]. The pitchfork bifurcations are the mechanism for the successive doubling of periods of stable orbits, and their accumulation point, an orbit of infinite period, is the so-called chaos threshold. The tangent bifurcation is a different mechanism linking periodic orbits with chaos in which intermittency is a precursor to periodic behavior [7]. Like in conventional equilibrium second order transitions, the bifurcation points have universal properties that can be obtained by means of renormalization group (RG) techniques, which for maps of the logistic type are based on functional composition and rescaling. As a result of this the *static* universal properties such as those derived from fixed-point maps and their perturbation equations are well understood [7]. However, the *dynamical* properties and their implications are only now being studied in a similar fashion by the same RG approach [8]. Here we present RG analytical results for the universal dynamics at bifurcation points of unimodal (unidimensional with a single maximum) maps of arbitrary nonlinearity $\zeta > 1$.

At each of the map bifurcation points the sensitivity to initial conditions ξ_t obeys power law behavior for large iteration time t in contrast to the usual exponential growth or decay generic of noncritical states [2]. This of course implies the vanishing of the Lyapunov λ_1 coefficient that measures either periodic ($\lambda_1 < 0$) or chaotic behavior ($\lambda_1 > 0$). The generalized nonextensive theory offers an attractive alternative to describe the dynamics at such critical points

[2], the sensitivity ξ_t would in these cases obey the q-exponential expression with a q-generalized Lyapunov coefficient λ_q

$$\xi_t = \exp_q(\lambda_q t) \equiv [1-(q-1)\lambda_q t]^{-\frac{1}{q-1}} \quad (q \in \mathbb{R}), \tag{1}$$

that yields the customary exponential ξ_t with λ_1 when $q \to 1$. The definition of λ_q differs from other generalizations of a Lyapunov coefficient [9]. One of us has recently pointed out [8] a clear connection between the fixed-point RG solutions at bifurcation points of unimodal maps and nonextensive entropy extremal properties. The aim of this letter is to work out analytical solutions for the dynamics and the sensitivity to initial conditions in the proximity of the first bifurcations of both types, and to corroborate these results numerically. In doing this we confirm previous estimations of the nonextensive parameter q and calculate for the first time the q-generalized Lyapunov coefficient λ_q in terms of the map parameters.

Let us start by briefly recalling the background formalism. We consider the generalization to nonlinearity of order $\zeta > 1$ of the logistic map

$$f_\mu(x) = 1 - \mu|x|^\zeta, \tag{2}$$

where $-1 \leq x \leq 1$ and $0 \leq \mu \leq 2$. These maps display a "pitchfork bifurcation regime" for $\mu < \mu_\infty$ where μ_∞ is the value of the period-doubling accumulation point (onset of chaos). On the other hand, as a consequence of the tangent bifurcations the "chaotic regime" for $\mu > \mu_\infty$ becomes interrupted at certain values of μ by windows of periodic behavior [7]. The solution of Hu and Rudnick [7, 10] to the Feigenbaum RG recursion relation [11] for the tangent bifurcation was obtained as follows. For the transition to a periodic window of order n there are n points for which the original map f is tangent to the line of unit slope. Choosing one of these points, shifting the origin of coordinates to this point, and making an expansion of the n-th composition of the original map, one obtains

$$f^{(n)}(x) = x + u|x|^z + o(|x|^z), \tag{3}$$

where u is the leading expansion coefficient. The RG fixed-point map $x' = f^*(x)$ was found to be

$$x' = x[1-(z-1)u \, \mathrm{sgn}(x)|x|^{z-1}]^{-\frac{1}{z-1}}. \tag{4}$$

This solution has a power-series expansion in x that coincides with Eq. (3) in the two lowest-order terms. In Ref. [8] it was observed that the previous scheme is also applicable to the pitchfork bifurcations $df^{(2^{k-1})}(x)/dx\Big|_{x=0} = -1$ of order $n = 2^k$, $k = 1, 2, ...,$ provided that the sign of u is changed for $x > 0$. As it will become clear below, we note that the power z appearing in Eqs. (3) and (4) is different from the power ζ in the original map Eq. (2).

Now, taking into account that the fixed-point map Eq. (4) satisfies $f^*(f^*(x)) = \alpha^{-1} f^*(\alpha x)$ with $\alpha = 2^{1/(z-1)}$, we obtain by repeated functional composition the following remarkable property

$$f^{*^{(m)}}(x) = \frac{1}{m^{\frac{1}{z-1}}} f^*(m^{\frac{1}{z-1}} x), \quad m = 1, 2, ... \tag{5}$$

This property implies that, for a total number of iterations $t = mn$, $m = 1, 2, ...$, with sufficiently small initial x_0, and for n and therefore t large, the fixed-point map can be written as

$$x_t \equiv \left[f^{(n)}\right]^{(m)}(x_0) = x_0[1 - (z-1)a \,\mathrm{sgn}(x_0)|x_0|^{z-1}t]^{-\frac{1}{z-1}}, \tag{6}$$

where $a \equiv u/n$. This is equivalent to a continuous time t approximation of Eq. (4) obtained by taking $x_t = x'$, $x_0 = x$ and $at = u$. We notice also that by introducing an additional index $l = 1, 2, ..., n-1$, the same expression holds for $x_{t+l} = f^{(mn+l)}(x_0)$, since x_0 on the right-hand side of this equation can be replaced by $f^{(l)}(x_0)$. In words, the time evolution of the original map in the neighborhood of a bifurcation point of order n, consists of a bundle of n orbits or trajectories of the form (6). But if one considers instead the time evolution of the n-composed map $f^{(n)}$ this is given by a single orbit of the form (6), only now with $t = m$ and $a = u$. It can also be verified that Eq. (6) satisfies the property $dx_t/dx_0 = (x_t/x_0)^z$, and this implies that the sensitivity to initial conditions $\xi_t \equiv \lim_{\Delta x_0 \to 0}(\Delta x_t/\Delta x_0)$ has the form

$$\xi_t(x_0) = [1 - (z-1)a \,\mathrm{sgn}(x_0)|x_0|^{z-1}t]^{-\frac{z}{z-1}}. \tag{7}$$

Comparison of Eq. (7) with Eq. (1) yields the identifications [12]

$$q = 2 - \frac{1}{z} \;\text{ and }\; \lambda_q(x_0) = za \,\mathrm{sgn}(x_0)|x_0|^{z-1}. \tag{8}$$

Further, by using the known [13] form $\rho(x) \sim |x|^{-(z-1)}$ for the invariant distribution of $f^{(n)}$ in Eq. (3) , we have that the average $\bar{\lambda}_q$ of $\lambda_q(x_0)$ over x_0 yields the piece-wise constant

$$\bar{\lambda}_q = za \,\mathrm{sgn}(x_0). \tag{9}$$

It is interesting to notice [8] that this average corresponds to the q-extension of the customary expression for the Lyapunov coefficient λ_1, obtained as the average of $\ln|df(x)/dx|$ over $\rho(x)$. Here we have

$$\bar{\lambda}_q = \int dx \rho(x) \ln_q \left|\frac{df^{(n)}(x)}{dx}\right|, \tag{10}$$

where $\ln_q(y) \equiv (y^{1-q} - 1)/(1-q)$, and $\ln_q\left(\exp_q(x)\right) = x$. Likewise, for the n-composed map $f^{(n)}$ the sensitivity to initial conditions is characterized by the

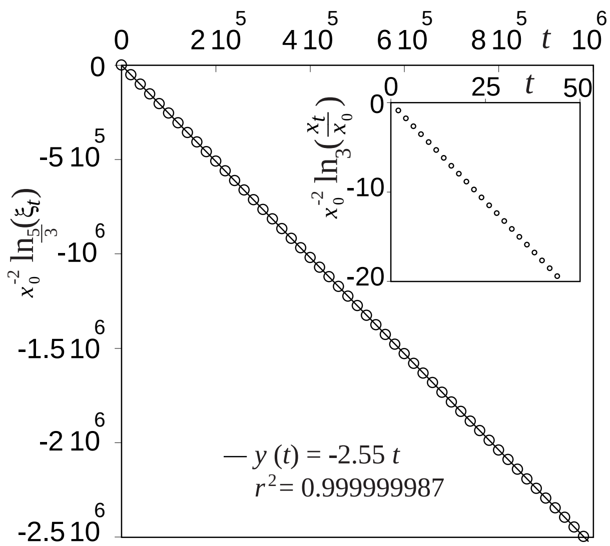


Figure 2: Weak insensitivity to initial conditions for the 1st pitchfork bifurcation when $\zeta = 1.75$. The straight line for the q-logarithm of ξ_t vs t with $q = 5/3$ and the slope $m = -2.55$ confirms the analytical result and the specific values predicted for q and λ_q for this transition. The inset shows the corresponding behavior for a single trajectory.

q-generalized Lyapunov coefficient $\overline{\Lambda}_q = n\bar{\lambda}_q = zu$. In Ref. [8] the alternative expression for the sensitivity to initial conditions was proposed

$$\xi_t = \left[\exp_q(\lambda_q t)\right]^q \equiv [1-(q-1)\lambda_q t]^{-\frac{q}{q-1}}, \tag{11}$$

as suggested by a noteworthy similarity between the RG perturbation expression for x_t and the corresponding nonextensive generalization of the Kolmogorov-Sinai entropy [8]. Using this form, the identifications are $q = z$ and $\bar{\lambda}_q = a$.

Next, we proceed to show explicit results for the index q and the q-generalized Lyapunov coefficient $\bar{\lambda}_q$ for the first pitchfork and tangent bifurcations. A pitchfork bifurcation of order $n = 2^k$ (see Fig. 1a) is characterized by the following conditions

$$\begin{aligned} f^{(2^{k-1})}(y_c) &= y_c \quad (k = 1, 2, ...), \\ \left.\frac{df^{(2^{k-1})}}{dy}\right|_{y=y_c} &= -1 \text{ and } \left.\frac{df^{(2^k)}}{dy}\right|_{y=y_c} = 1, \end{aligned} \tag{12}$$

these determine the values of the critical parameter μ_c and of the 2^{k-1} critical

positions y_c. Shift of coordinates ($x \equiv y - y_c$ and $x' = f^{(2^k)}(y) - y_c$) and expansion yields

$$f^{(2^k)}(x) = x + u|x|^3 + o(x^3), \qquad (13)$$

since $d^2 f^{(2^k)}/dx^2|_{x=0} = 0$ always at these transitions, independently of the nonlinearity ζ. The coefficient u is then given by $u = \pm(1/6)d^3 f^{(2^k)}/dx^3|_{x=0}$, where the $+$ sign applies for $x > 0$ and the $-$ sign for $x < 0$, and where $d^3 f^{(2^k)}/dx^3|_{x=0} < 0$. As a first result we have that $z = 3$ and $q = 5/3$ for the sensitivity to initial conditions of all the pitchfork bifurcations of any nonlinearity ζ. Note that for any value of $x_0 \neq 0$ the argument of the q-exponential (7) is negative, and this, combined with $q > 1$, implies that the pitchfork bifurcations display *weak insensitivity* [2] to initial conditions at both sides of the critical position. In the case of the first pitchfork bifurcation ($k = 1$) its location is given by $\mu_c = (\zeta + 1)^\zeta/(\zeta^\zeta(\zeta + 1))$ and $y_c = \zeta/(\zeta + 1)$, whereas the coefficient u is given by $u = \mp(\zeta^4 + 2\zeta^3 - 2\zeta - 1)/(6\zeta^2)$. So that the ζ-dependent q-generalized Lyapunov coefficient for $f^{(2)}$ is

$$\bar{\lambda}_q = -\frac{1}{2}\frac{\zeta^4 + 2\zeta^3 - 2\zeta - 1}{\zeta^2}. \qquad (14)$$

In Fig. 2 we present the numerical results for the first pitchfork bifurcation with $\zeta = 1.75$ that corroborate the analytical prediction $\bar{\lambda}_q = -2.5466$. We have verified Eq. (14) for several values of ζ.

The location and the expansion coefficient for the second pitchfork bifurcation ($k = 2$) for general ζ are harder to obtain analytically, but as an illustration we give results for the logistic map ($\zeta = 2$). We obtain $\mu_c = 5/4$ and the two critical positions $y_c = 2(1 \pm \sqrt{2})/5$. Interestingly, $\bar{\lambda}_q$ for the first critical position is $\bar{\lambda}_q = -3(250 + 125\sqrt{2})/4$, while for the second position is $\bar{\lambda}_q = -3(250 - 125\sqrt{2})/4$.

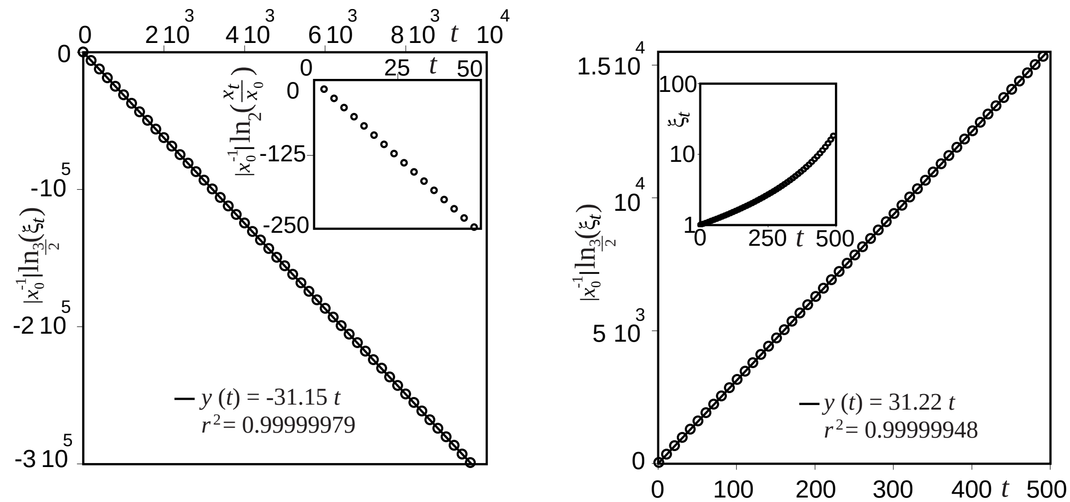
The conditions that determine the critical parameter and the n critical positions of a tangent bifurcation of order n (see Fig. 1b) are

$$f^{(n)}(y_c) = y_c \ \text{ and } \ \left.\frac{df^{(n)}}{dy}\right|_{y=y_c} = 1, \qquad (15)$$

so that, by carrying out the same change in coordinates as before, one obtains the expansion

$$f^{(n)}(x) = x + u|x|^2 + o(x^2), \qquad (16)$$

where the coefficient u is now given by $u = (1/2)d^2 f^{(n)}/dx^2|_{x=0} > 0$. Hence we have $z = 2$ and $q = 3/2$ for the sensitivity to initial conditions of all the tangent bifurcations regardless of the value of ζ. The left-hand side of the tangent bifurcation points exhibits, in analogy to the previous case, a *weak insensitivity* to initial conditions. However at the right-hand side of the bifurcation the argument of the q-exponential becomes positive and this, together with $q > 1$,



a) b)

Figure 3: a): Weak insensitivity to initial conditions at the left-hand side of the 1st tangent bifurcation when $\zeta = 2$. The straight line for the q-logarithm of ξ_t vs t with $q = 3/2$ and the slope $m = -31.15$ confirms the analytical result and the specific values predicted for q and λ_q for this transition. The inset shows the corresponding behavior for a single trajectory. b): 'Super strong' sensitivity to initial conditions at the right-hand side of the 1st tangent bifurcation when $\zeta = 2$. The straight line for the q-logarithm of ξ_t vs t with $q = 3/2$ and the slope $m = 31.22$ confirms the analytical result and the specific values predicted for q and λ_q for this transition. The inset shows the logarithm of ξ_t vs t where faster than exponential growth can be clearly observed.

results in a *'super strong' sensitivity* to initial conditions, i.e. a sensitivity that is faster than exponential. Once again we use the logistic map $\zeta = 2$ as an illustration, and we choose for its first tangent bifurcation ($n = 3$) at $\mu_c = 7/4$ one of the three critical positions, $y_c = 0.031405...$, and determine its expansion coefficient to be $u = 15.608...$. Thus for $f^{(3)}$ the q-generalized Lyapunov coefficient is $\bar{\lambda}_q = \pm 31.216...$ for $x_0 \gtrless 0$. In Fig. 3a) we plot the numerical results for the left side of this bifurcation, while in Fig. 3b) we show the 'super strong' sensitivity to initial conditions characteristic of the right side of the tangent bifurcation.

Naturally, only for very low order n of the bifurcation and/or special values of the nonlinearity ζ it is feasible to obtain algebraic solutions for the locations of the transitions and then for the leading expansion coefficient u. Nevertheless, it is always possible to implement numerical methods to determine them, and consequently the q-generalized Lyapunov coefficient $\bar{\lambda}_q$. As shown in Figs. 2 and 3 the *a priori* numerical calculations for the simplest examples selected here provide a striking confirmation of our RG predictions for both q and $\bar{\lambda}_q$.

In summary, we have fully determined the dynamical behavior at the pitchfork and tangent bifurcations of unimodal maps of arbitrary nonlinearity $\zeta > 1$. This was accomplished via the consideration of the solution to the Feigenbaum RG recursion relation for these types of critical points. Our study has made use of the specific form of the ζ-logistic map but the results have a universal validity as conveyed by the RG approach. The RG solutions are exact and have the analytical form of q-exponentials, we have shown that they are the time (iteration number) counterpart of the static fixed-point map expression found by Hu and Rudnick for the tangent bifurcations and that is applicable also to the pitchfork bifurcations [8]. The q-exponential form of the time evolution implies an analytical validation of the expression for the sensitivity to initial conditions suggested by the nonextensive statistical mechanics [2]. It also provides straightforward predictions for q and $\bar{\lambda}_q$ in terms of the map critical point properties [8]. We found that the index q is independent of ζ and takes one of two possible values according to whether the transition is of the pitchfork or the tangent type. The q-generalized Lyapunov coefficient $\bar{\lambda}_q$ is simply identified with the leading expansion coefficient u. As we have shown these predictions are unquestionably corroborated by *a priori* numerical calculations. Significantly, both families of bifurcations display non-canonical dynamical behavior (either weak insensitivity or 'super strong' sensitivity to initial conditions). The universal attribute of our results invites some reflection on the rationale for non-canonical dynamics. This is most apparently linked to the fact that the tangency shape of the map at these critical points either effectively confines or expels trajectories causing an abnormal, incomplete, sampling of phase space (here $-1 \leq x \leq 1$).

Acknowledgments

We would like to thank C. Tsallis and L.G. Moyano for useful discussions and comments. AR gratefully acknowledges the hospitality of the Centro Brasileiro de Pesquisas Físicas where this work was carried out and the financial support given by the CNPq processo 300894/01-5 (Brazil). AR was also partially supported by CONACyT grant 34572-E and by DGAPA-UNAM grant IN110100 (Mexican agencies). FB has benefitted from partial support by CAPES, PRONEX, CNPq, and FAPERJ (Brazilian agencies).

References

[1] C. Tsallis, J. Stat. Phys **52**, 479 (1988); E.M.F. Curado and C. Tsallis, J. Phys. A **24**, L69 (1991) [Corrigenda: **24**, 3187 (1991) and **25**, 1019 (1992)]. A recent review can be found in C. Tsallis, in *Nonextensive Statistical Mechanics and Its Applications*, eds. S. Abe and Y. Okamoto, Lecture Notes in Physics **560**, 3 (Springer, Berlin, 2001). See http://tsallis.cat.cbpf.br/biblio.htm for full bibliography.

[2] C. Tsallis, A.R. Plastino and W.-M. Zheng, Chaos, Solitons and Fractals **8**, 885 (1997). See also P. Grassberger and M. Scheunert, J. Stat. Phys. **26** 697 (1981), T. Schneider, A. Politi and D. Wurtz, Z. Phys. B **66** 469 (1987), G. Anania and A. Politi, Europhys. Lett. **7** 119 (1988) and H. Hata, T. Horita and H. Mori, Progr. Theor. Phys. **82** 897 (1989).

[3] U.M.S. Costa, M.L. Lyra, A.R. Plastino and C. Tsallis, Phys. Rev. E 56, 245 (1997).

[4] M.L. Lyra and C. Tsallis, Phys. Rev. Lett. 80, 53 (1998).

[5] F.A.B.F. de Moura, U. Tirnakli and M.L. Lyra, Phys. Rev. E 62, 6361 (2000).

[6] F. Baldovin, C. Tsallis and B. Schulze, cond-mat/0203595.

[7] H.G. Schuster, *Deterministic Chaos. An Introduction*, 2nd Revised Edition (VHC Publishers, Weinheim, 1988).

[8] A. Robledo, cond-mat/0202095.

[9] C. Beck and F. Schlogl *Thermodynamics of chaotic systems: an introduction* (Cambridge University Press, Cambridge, UK, 1993), pages 164ff.

[10] B. Hu and J. Rudnick, Phys. Rev. Lett. **48**, 1645 (1982).

[11] M. Feigenbaum, J. Stat. Phys. **19**, 25 (1978); **21**, 669 (1979).

[12] M. Buiatti, P. Grigolini and A. Montagnini, Phys. Rev. Lett. **82**, 3383 (1999).

[13] P. Gaspard and X.J. Wang, Proc. Natl. Acad. Sci. USA, **85**, 4591 (1988).